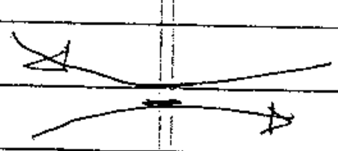
- Simple Ideas in Non-Ideal MHD

→ Freezing-in law:

$$\frac{d}{dt}\underline{B} = \underline{B}\cdot\nabla\underline{v} + \eta\nabla^2\underline{B}$$

↑ breaking → small scale → singularity / turbulence

→



→ current sheet
singular layers

→ sites of reconnection → boundary layer problem

→ topology change

→ So

- Sweet-Parker Reconnection theory

- Re-visiting magnetic helicity; Taylor Theory

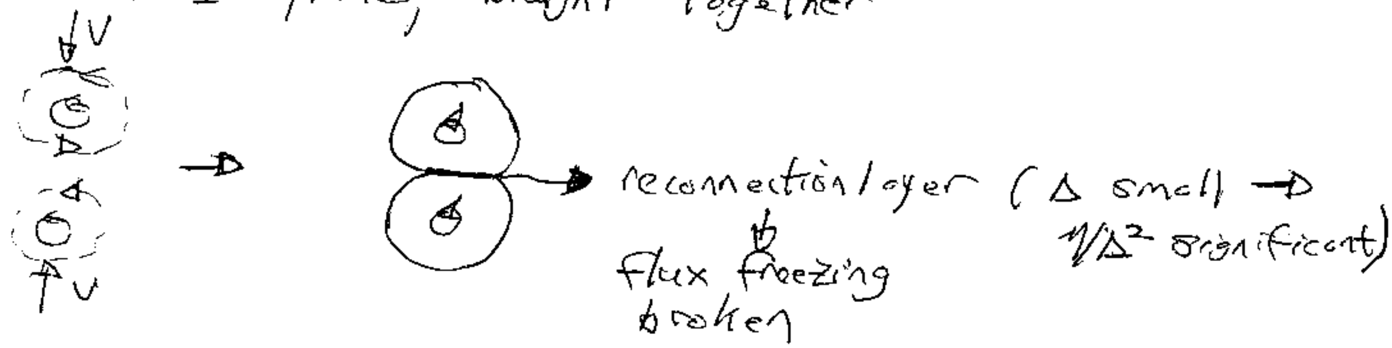
- anomalous resistivity (again)

- flux expulsion

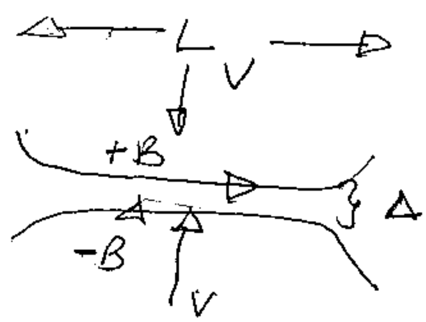
→ Breakdown of Flux Freezing – Magnetic Reconnection ?!

Simple Example : Sweet – Parker Problem (re-visit later)

→ consider two cylinders of plasma carrying current ⊥ plane, brought together



⇒ consider layer



current sheet

$\Delta < L$

2 plasma slabs brought together at V

What Happens?

Stationary Solution Possible?

$\nabla\cdot\underline{V}=0$ $\qquad \frac{d}{dt}\underline{B} = \underline{B}\cdot\nabla\underline{V} + \eta\nabla^2\underline{B}$

$$S_{ij}^{(0)} = \begin{pmatrix} 0, & 0 \\ 0, & -V\delta(y) \end{pmatrix}$$

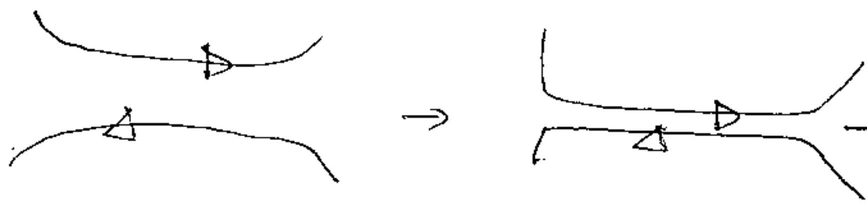
S → rate-of-strain tensor

$\delta(y)$ → singularity!

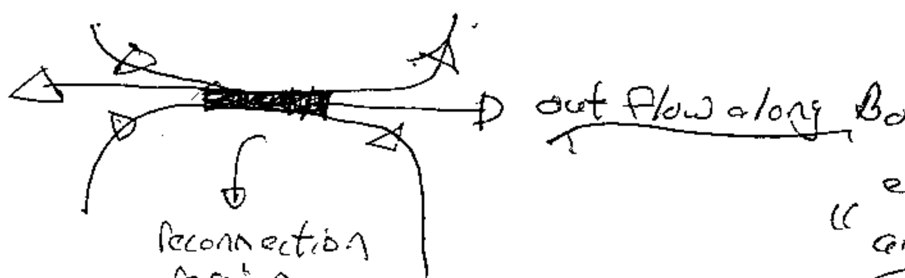
tip-off of small scale generation in $\underline{B}$
⇒ resistive diffusion, breaking of freezing-in ...

i.e. for stationary solution,

$$-\frac{\underline{B}\cdot\underline{\nabla}\,\underline{V}}{\eta} = \nabla^2 \underline{B} \,, \qquad \underline{\nabla}\cdot\underline{V} = 0$$

$\Rightarrow$ 

$\to$ Δ small
$\nabla(B^2/8\pi)$ increasing
fluid expelled

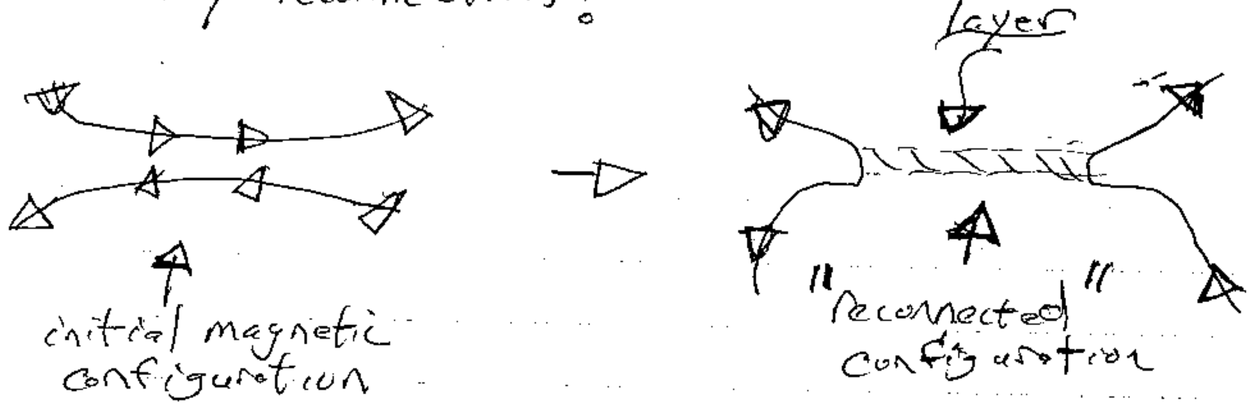
$\Rightarrow$ 

(large, resistive dissipation)

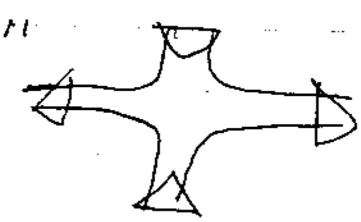
"expect opposed fields "annihilate""

N.B. A particular V value is required for stationarity

N.B. $\to$ why "reconnection"?



$\to$ Flow is "stagnation point"



→ How Calculate ? → Match In-Flow → Out-Flow
(S-P. is a great Back-of-Envelope ...)

Conserved : ① – mass $(\underline{\nabla}\cdot\underline{V} = 0)$

② – momentum in $\hat{x}$ direction (symmetry)

③ – energy balance, →
⇝ rate of field 'delivery' to reconnection region
MUST BALANCE
⇝ rate of ohmic dissipation $E\cdot J \sim \eta J^2$

(1)

$$\rho_0 \underset{\text{inflow}}{V} \overset{\text{extent in } \hat{x}}{L} = \rho_0 \underset{\text{outflow}}{V_0} \overset{\text{extent in } \hat{y}}{\Delta} \qquad VL = V_0 \Delta$$

$$V = V_{out}\,\Delta / L$$

②

$$\rho_0\left(\frac{\partial \underline{V}}{\partial t} + \underline{V_0}\cdot\nabla\underline{V}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{V}\times\underline{\omega}$$

symmetry :

$$0 = \underline{\nabla}\left(P + \frac{B^2}{8\pi} + \frac{\rho_0 V^2}{2}\right)$$

modified Bernoulli Eqn.

$\underline{a}$ $\underline{b}$

$\underline{a} \to V = 0, \quad B$ finite

$\underline{b} \to V = V_{out}, \quad B \to 0$

$$\left(\frac{B^2}{8\pi} \ll P\right)$$

So $P + \frac{B^2}{8\pi} + \rho_0 \frac{V^2}{2} = \text{const.}$

$$P + \frac{B^2}{8\pi} = P + \rho_0 \frac{V_{out}^2}{2}$$

$$V_{out}^2 = B^2/4\pi\rho_0 = V_A^2$$

↳ Alfven speed

$V_{out} = V_A$

$V = V_A \frac{\Delta}{L}$

specifies "speed V", in terms Δ.

③ 

Energy balance

⇒

(Rate of Magnetic Energy Inflow) = (Rate of Ohmic Dissipation, net)

$$P_{OH} = \frac{J^2}{\sigma}\Delta L \text{ so } \dot{E}_{OH} = \frac{J^2}{\sigma} L\Delta$$

$$= \left(\frac{c}{2\pi}\right)^2 \frac{B^2}{\Delta^2}\frac{L\Delta}{\sigma}$$

$\nabla \times B = \frac{4\pi}{c} J$

$2B = \frac{4\pi}{c} J \Delta$

$$P_{in} = 2\left(B^2/8\pi\right) VL = \dot{E}_{in}$$

balance $\Rightarrow$ $\cancel{2}\left(\frac{\cancel{B^2}}{\cancel{8\pi}}\right) V \cancel{L} = \frac{c^2}{\cancel{4\pi}\sigma}\frac{\cancel{B^2}}{\Delta^2}\cancel{L}\Delta$

$$V = \left(\frac{c^2}{4\pi\sigma}\right)/\Delta \sim \frac{\eta}{\Delta}$$

$$\frac{c^2}{4\pi\sigma} \equiv \eta \left(\sim \frac{L^2}{T}\right)$$

$$V = V_A \, \Delta / L$$

$$V = \eta / \Delta \quad \Rightarrow \quad \frac{\Delta}{L} = \left(\frac{\eta}{L V_A} \right)^{1/2} = \left(1 / R_m \right)^{1/2}$$

and $\quad V = V_A / \sqrt{R_m}$

$R_m = \dfrac{V L}{\eta} \equiv$ Magnetic Reynolds # (here with $V = V_A$)

$\Rightarrow$ Punch Line: (for large R_m)

① – layer is thin $\quad \dfrac{\Delta}{L} \sim 1/\sqrt{R_m}$

– speed is faster than η/L, slower than V_A $\quad V \sim V_A / \sqrt{R_m}$

② flow pattern is a/a' stagnation $\Rightarrow$ ejection from reconnection layer at V_A

Moral of this story:

$\rightarrow$ freezing-in violated when flows bring opposing $\underline{B}$ into contact $\rightleftarrows$.

$\Rightarrow$

$\rightarrow$ generates singularities $\Rightarrow$ thin current layers, which alter initial magnetic topology

$\Rightarrow$ "magnetic reconnection", "tearing", etc.

→ Magnetic Helicity

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

{ V is taken to be the volume of a 'flux tube'.

- what, yet another invariant!?!...

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; A \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

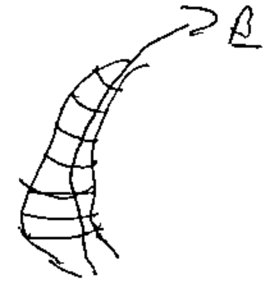
Proceed via:
- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x\, \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x\, \underline{\nabla}\cdot(\underline{B}\,\chi)$$



$= 0$, to surface term. $\left\{ \underline{B}\cdot\hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\frac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V}\times\underline{\nabla}\times\underline{A} - c\nabla\phi - c\eta\underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V} + \eta\nabla^2\underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt}\int_V d^3x\, (\underline{A}\cdot\underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt}\cdot\underline{B} + \underline{A}\cdot\frac{d\underline{B}}{dt}\right) + \int \underline{A}\cdot\underline{B}\,\frac{d}{dt}\,d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V} \cdot \underline{\nabla} \underline{A}) \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{A} \cdot (\underline{V} \cdot \underline{\nabla} \underline{B}) + \underline{A} \cdot \underline{B} \; \underline{\nabla} \cdot \underline{V} \right)$$

where $\frac{d}{dt} d^3x = \underline{\nabla} \cdot \underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma} \cdot d\underline{\ell} + d\underline{\sigma} \cdot \frac{d}{dt} d\underline{\ell}$

$$= - d\underline{\ell} \cdot \underline{\nabla} \underline{V} \cdot d\underline{\sigma} + (\underline{\nabla} \cdot \underline{V})(d\underline{\sigma} \cdot d\underline{\ell}) + d\underline{\ell} \cdot \underline{\nabla} \underline{V} \cdot d\underline{\sigma}$$

$$= \underline{\nabla} \cdot \underline{V} \, d^3x$$

s.t. and $\underline{B} \cdot \hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\left(\underline{B} \cdot \cancel{\underline{V} \times \underline{B}} - c \, \cancel{\underline{B} \cdot \underline{\nabla} \phi} - c \eta \, \underline{J} \cdot \underline{B} \right) + \underline{A} \cdot \left(\underline{\nabla} \times (\underline{V} \times \underline{B}) \right) + \underline{\nabla} \cdot \left((\underline{A} \cdot \underline{B}) \underline{V} \right) + \underline{A} \cdot \eta \nabla^2 \underline{B} \Big]$$

where $\underline{A} \cdot (\underline{V} \cdot \underline{\nabla} \underline{B}) + \underline{B} \cdot (\underline{V} \cdot \underline{\nabla} \underline{A}) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} = \nabla \cdot (\underline{V} \, \underline{A} \cdot \underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla \cdot \left((\underline{A} \cdot \underline{B}) \underline{V} \right) + \underline{\nabla} \cdot \left((\underline{V} \times \underline{B}) \times \underline{A} \right) + \cancel{(\underline{V} \times \underline{B}) \cdot (\underline{\nabla} \times \underline{A})} - c\eta \, \underline{\nabla} \cdot \underline{B} - \eta (\underline{A} \cdot \underline{\nabla} \times \underline{J}) c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} \right.\right.$$

$$\left.\left. + c\eta(\underline{A}\times\underline{J}) \right] - c\eta\,\underline{J}\cdot\underline{B} - c\eta\,\underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S} \cdot \left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\,\underline{A}\times\underline{J} \right]$$

$$- 2\int d^3x \left[c\eta\,\underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S} \cdot \left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \underbrace{(\underline{A}\cdot\underline{v})\underline{B}}_{0} \right] - c\eta \int d\underline{S} \cdot \underline{J}\times\underline{A}$$

$\underline{B}\cdot\underline{n} = 0$, on tube

$$- 2c\eta \int d^3x (\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S} \cdot \left[\nabla\underline{B}\cdot\underline{A} - \cancel{\underline{A}\cdot\nabla\underline{B}} \right] - 2c\eta \int d^3x\, \underline{J}\cdot\underline{B}$$

$$= -2c\eta \int d^3x (\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta \int d^3x\, (\underline{J}\cdot\underline{B})}$$

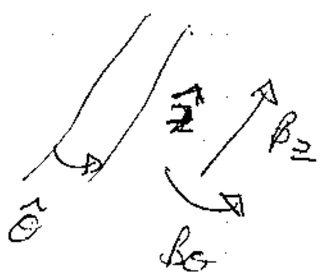
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$
(non-singular $\underline{J}$)

$\therefore$ Magnetic Helicity is conserved in ideal MHD.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $q(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R \mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{r B_z}$ → field line pitch.

(length scale over which winding varies)

cylindrical plasma ⇒ $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r} \int_0^r r \, B_z \, dr$

$A_z = - \int_0^r B_\theta \, dr$

so $\quad \underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\,dr - B_z\int_0^r B_\theta\,dr$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\,dr - B_z \int_0^r B_\theta\,dr$$

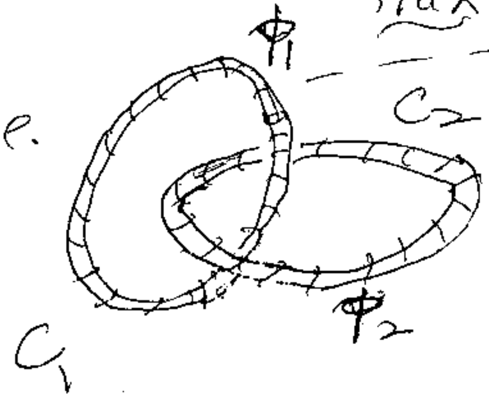
$$\underline{A}\cdot\underline{B} = B_z\left[\mu\int_0^r \frac{B_\theta}{\mu}\,dr - \int_0^r B_\theta\,dr\right]$$

$= 0$ for constant μ

$\therefore$ non-zero helicity requires $\mu = \mu(r)$
i.e. – pitch varies with radius
$\Rightarrow$ $\boxed{\text{magnetic shear}}$

– physically $\rightarrow$ helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux $\Phi = \underbrace{\int d\underline{A}}_{\text{x-section area}}\cdot\underline{B} = \underbrace{\phi_1}_{\text{const}}$

tube 2: $\Phi = \phi_2$

field in loops, only

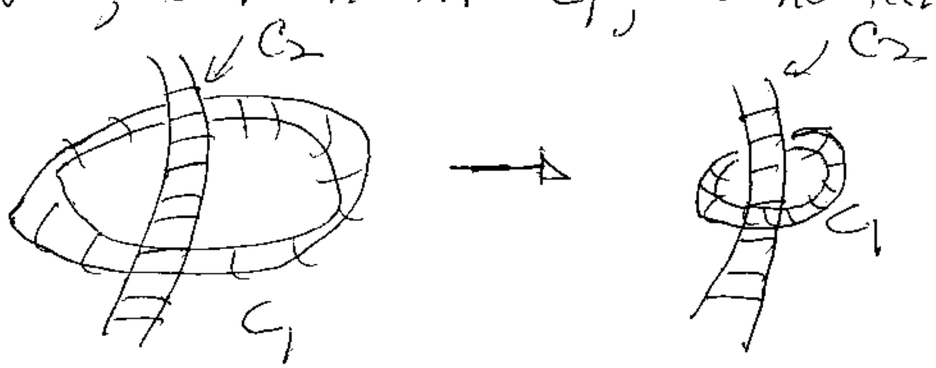
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \underbrace{\oint_{C_1} d\ell}_{\text{along loop}} \underbrace{\int_{A_1} \underline{dS}}_{\text{X-sect area}}\, A\cdot\underline{B}$$

$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

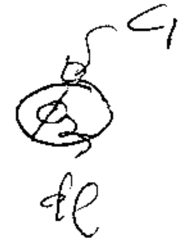
$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



re-oriented

$\Rightarrow$ in X section:



but $$\int_{C_1} A\cdot d\ell = \int_{A\ \text{enclosed}} B\cdot dS = \Phi_2$$

so . . . $K_1 = \Phi_1 \Phi_2$ $\Rightarrow$ product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

$\therefore \quad K = 2\Phi_1 \Phi_2$

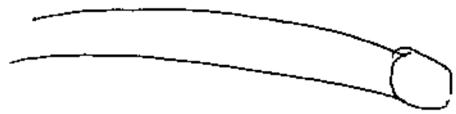
if n windings $\quad K = K_1 + K_2 = \pm 2n\Phi_1 \Phi_2$.

$\Rightarrow$ helicity is measure of self-linkage of magnetic configuration.

Why care $\rightarrow$ Taylor Conjecture (1974) (J. B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP



$\sim$ toroid

$\sim$ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$

$\Rightarrow$ force free

$\Rightarrow$ why so robust?, especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to η
- toroidal plasma $\Leftrightarrow$ many small tubes

etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but, global helicity conserved.

i.e.

$$\int_{\text{plasma volume}} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow \text{conserved}.$$

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay
- ↗ energy cascades → small scale
- ↘ helicity cascades → large scale (less dissipation)

– relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..